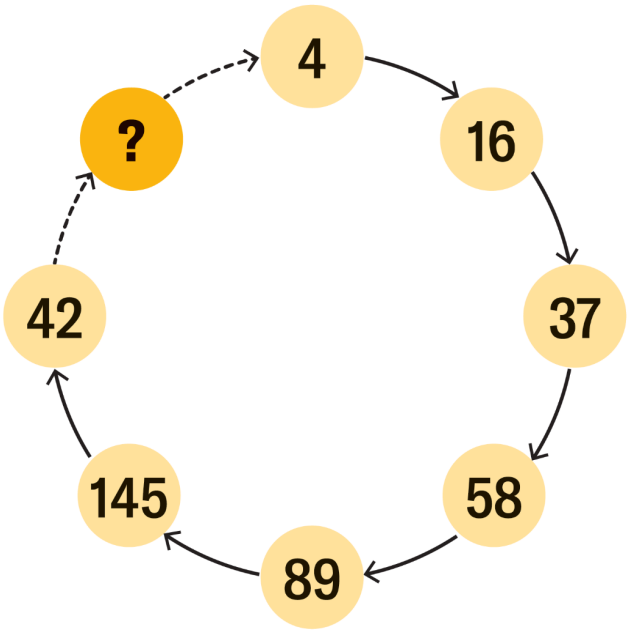
Benjamin Alperovich

28 May 2026

Fixed Points and Cycles in Iterated Digit Power Functions



I. Research Question: Define $f(n, k)$ as the sum of the kth powers of the digits of n.

Example: $f(32, 2) = 3^2 + 2^2 = 13$. Now repeatedly apply this function to a value and see its behavior. For a fixed exponent k in base 10, how do the number and lengths of cycles behave under the iteration of the function $f(n, k)$?

II. Introduction:

Many popular problems in number theory consider the digits of numbers, and many others consider powers of these numbers. This research paper connects these two concepts, exploring what happens when you take the sum of the powers of the digits of the number. The goal of this investigation is to explore what happens when we define a function that returns the sum of the powers of the digits of the number, and iterate this function over and over again. This creates cycles, and the output eventually repeats. The goal is primarily to analyze these cycles—the length, the size, and the number of cycles. A large part of this paper is devoted to analyzing fixed points: numbers that when the function is applied to it returns the number itself. For example: $1^3 + 5^3 + 3^3 = 153$. I analyze which exponents have fixed points, and how we can narrow down which numbers could potentially be fixed points through modular arithmetic and other methods. I used upper bound arguments to prove the fact that every function $f(n, k)$ eventually enters a cycle and to find things about where fixed points can exist. The idea of doing a problem revolving around cycles came to me when thinking about the collatz conjecture, and how it is unsolved in terms of whether or not every starting point enters the cycle of 1, 4, 2. I also considered math problems that explore the sums of the digits of numbers, and decided to try to find a cycle type problem in this area of math. I investigated happy numbers, which connect to the $k = 2$ case of my question and go in depth about how changing the base impacts the problem.

III. History:

This problem's fixed point scenario is a specific case of the Waring problem, which asks if for every natural number there exists a number such that every natural number is the sum of at most n-th powers of natural numbers. This was proven true by Hilbert in 1909. However, the modification is that the numbers exponentiated can only be digits, and those digits must be part of the original number itself. I did not find a complete classification of this exact iteration problem, so this paper focuses on computational and theoretical exploration rather than a complete solution. The study of cycles is a large area in graph theory, with a cycle usually defined as a path that comes back to the original vertex. However, the definition of cycle I choose includes coming back to any of the previously visited vertices. The iterative process of my question is also a widely known object of investigation, and the consequences of applying seemingly elementary operations can be quite unpredictable, as seen in the collatz conjecture and in my question. The investigation of sums of powers somewhat connects to fermat's last theorem, though it is inapplicable for any digit lengths past 2. My question is more related to the study of periodic functions, as for a cycle length of 2, for example, $f(f(n,\ k),\ k)\ =\ n$. In this paper, I explore whether repeated iteration of $f(n, k)$ eventually becomes periodic.

IV. Mathematical Background:

Define the function $f(n, k)$ as the sum of the kth powers of the digits of n. This function will be iterated throughout the paper to study its behavior. For example, if $n\ =\ 146$ and $k\ =\ 3$, then $f(146,\ 3)\ =1^3+4^3+6^3=\ 281$. Applying the function again, $f(281, 3)\ =2^3+8^3+1^3=\ =\ 521$. Repeated application of the function will be denoted by iteration, such as $f(f(n, k), k)$ representing multiple steps. For the majority of this paper, a mathematical background of algebra II is needed. This includes the composition of functions, which shows up as iterations of my

function. One should also know how to complete the square since I use it during my algebraic investigation of fixed points. The knowledge of place value of a number and permutations of those digits is also a subtopic of my paper where I note that the order of the digits does not matter. I also discuss lower and upper bounds to assess when fixed points can occur, so it is useful to know the minimum and maximum values of a d digit number to be 10^{d-1} and $10^{d} - 1$, respectively. I will also use properties of exponential growth in my paper, so it is important to understand how different functions compare in growth if both have an exponential term. I will use this to computationally search for a restriction when it comes to which exponents k can have fixed points. I will use modular arithmetic to rule out some integers that cannot be considered fixed points due to modular constraints. This includes knowing how residues behave when exponentiated, and the addition of multiple residues in a specific modulus. For example, when working in $(mod\ 4)$, any integer is congruent to $0, 1, 2, 3$, squaring these gives $0,\ 1,\ 0,\ 1\ (mod\ 4)$, respectively, which will later be used to rule out certain possibilities for fixed points. One should also understand what I define a cycle to be: a sequence of numbers which are separated by iterations of the function which loops around eventually, and a fixed point is a cycle length of 1, where it iterates back to itself. For example, for the exponent $k = 2$, a cycle is: $89,\ 145,\ 42,\ 20,\ 4,\ 16,\ 37,\ 58$, at which point the cycle stops because $5^2 + 8^2 = 89$, which has been hit before. Additionally, for proofs in my paper, one should know the pigeonhole principle, which states that if n items (pigeons) are placed into m containers (holes), and $n > m,$ then at least one container must hold more than one item. This is a foundational concept that I will use to prove that every starting point eventually enters a cycle. It is also important to understand what is meant by the density with which a starting point goes to a specific cycle. For example, if there is only one cycle for a given exponent k, then the density for

that cycle would be 1, as every starting point is eventually forced into that one cycle. On the other hand, if there are numerous distinct cycles that a starting point could enter for a given exponent k, then the density for each cycle would be the fraction of the total starting points that go to that cycle. These concepts serve as the foundation for analyzing how the function behaves when iterated, determining when cycles occur, and restricting fixed points.

V. Investigation:

Let us first start by examining some cases of growth. Through inspection, two digit numbers with large digits end up having an output that is higher than the initial value. For example, $f(86, 2) = 8^2 + 6^2 = 100$. Note that after applying f a second time, $f(100, 2) = 1$, which is a fixed point because no matter how many times you apply this function, the input and output are equal to each other.

Note: 1 is always a fixed point for every exponent, so it will be disregarded in search of others. It is also worth noting that $f(68, 2)$, which is a permutation of 86, yields the same output as $f(86, 2)$. Therefore, all outputs of permutations are going to be equal.

Algebraic Constraints on Fixed Points

Now we investigate fixed points.

Consider a one-digit positive integer A. A must satisfy $A^2 = A$ in order to be a fixed point, and the only valid solution is $A = 1$.

Consider a two digit integer AB. If it is a fixed point, it must satisfy

$A^2 + B^2 = 10A + B$, $A^2 - 10A + 25 - 25 + B^2 - B + \frac{1}{4} - \frac{1}{4} = 0$,

$(A - 5)^2 + (B - \frac{1}{2})^2 = \frac{101}{4}$, $(2A - 10)^2 + (2B - 1)^2 = 101$. Through inspection, the only perfect square pair that adds to 101 is 100 and 1, which can't occur because $2A - 10$ is

even and can't ever be 10 because A is a digit from 1 to 9, so $(2A - 10)^2$ achieves a maximum of 64 at $A = 9$. Therefore, no two digit fixed points exist for $k = 2$.

Investigating three digits,

$A^2 + B^2 + C^2 = 100A + 10B + C,$

$(A - 50)^2 - 2500 + (B - 5)^2 - 25 + (C - \frac{1}{2})^2 - \frac{1}{4} = 0,$

$(A - 50)^2 + (B - 5)^2 + (C - \frac{1}{2})^2 = 2525.25,$

$(2A - 100)^2 + (2B - 10)^2 + (2C - 1)^2 = 10101$. Consider $A = 1$, which is the smallest case for A, giving $(2B - 10)^2 + (2C - 1)^2 = 300$. This equation is impossible because the maximum value of $(2B - 10)^2$ is even and $(2C - 1)^2$ must be odd, and the sum of opposite parities can never be even. In the next smallest case $A = 2$, $(2B - 10)^2 + (2C - 1)^2 = 497$, which is also impossible because the maximum value of $(2B - 10)^2$ is 100, and the maximum value of $(2C - 1)^2$ is 289, which doesn't reach the desired 497.

With four digits, you have $A^2 + B^2 + C^2 + D^2 = 1000A + 100B + 10C + D$, and upon inspection this simply grows too large, as the maximum of sum of the squares is $4 * 81 = 324$, which does not reach the lowest A value that reaches 1000. The same applies to any higher digit numbers, since the exponential coefficient of A outgrows the maximal value of the sum of the squares of the digits. This leaves the only single digit fixed point for exponent $k = 2$ to be 1.

Let us now examine fixed points for the exponent $k = 3$.

For a 1 digit number, A, $A = A^3$, which has positive integer solutions of 1.

For a two digit number, $10A + B = A^3 + B^3$. Through inspection, the only case that could potentially work is when $A = 3$, because $A = 2, 4$ both go out of range since B is restricted to $0 - 9$. So, $30 + B = 27 + B^3$, $B + 3 = B^3$, which has no integer solutions.

For 3 digits, $A^3 + B^3 + C^3 = 100A + 10B + C$. This has numerous solutions, as shown in the table, such as (A, B, C): $\{(1, 5, 3),\ (3, 7, 1),\ (3, 7, 0), (4, 0, 7)\}$

For 4 digits, $A^3 + B^3 + C^3 + D^3 = 1000A + 100B + 10C + D$, the maximum value of the sum of the cubes is $4 * 9^3 = 2916$. But, this forces the other side to 9999, so in order to find values that are closer, the maximum when $A = 1$ is $1 + 3 * 9^3 = 2188$, which is now in range since the other digit can be decreased to make the number to be in the thousands. However, this devolves into a guess and check structure, and through computational methods, a 4 digit fixed point does not exist for $k = 3$. For higher digits, the maximum of the sum of the cubes is outgrown by the $10^{d-1}A$ term, so this guarantees no other fixed points exist for this exponent.

For $k = 4$:

1 digit: $A = A^4$, $A = 1$.

2 digit: $10A + B = A^4 + B^4$, A must be 2 to satisfy this equation, and $B + 4 = B^4$ has no integer solutions.

3 digit: $100A + 10B + C = A^4 + B^4 + C^4$

4 digit: $1000A + 100B + 10C + D = A^4 + B^4 + C^4 + D^4$

5 digit: $10000A + 1000B + 100C + 10D + E = A^4 + B^4 + C^4 + D^4 + E^4$.

The maximum value of the sum of these 4th powers is $5 * 9^4 = 32805$, so A is already limited to 3 and below. But, to achieve this maximum value, A must be 9, so when $A = 3$, the maximum

value is $3^4 + 4 * 9^4 = 26325$. So now, A is bounded to 2 or less. If $A = 2$, then the sum of the 4th powers must reach 20000, in order to achieve this, at least 2 of the other digits must be 9s, since that would make the sum 13122. If another digit is a 9, it would push the sum to 19683, at which point any integer greater than or equal to 5 satisfies this constraint. If another digit is an 8, then the last digit can only be an 8 or a 9 to potentially satisfy this equation. So these types of constraints exist for these kinds of equations.

After this, the $10^{d-1}A$ term outgrows the sum of the fourth powers.

Eventual Cycling

Theorem 1: For any integer n and exponent k, repeated applications of $f(n, k)$ eventually enters into a cycle.

For context, a cycle is defined as something that returns to a number that it has hit before. For example, for starting value 34: $34, 25, 29, 85, 89, 145, 42, 20, 4, 16, 37, 58, 89$. This ends because 89 has been hit before and it enters a cycle from there.

Proof: For a d digit number n, the largest output $f(n, k)$ could give is when all the digits of the number are 9s. So, the largest output would be $d * 9^k$. Additionally, any d digit number is at least 10^{d-1}. Since 10^{d-1} eventually outgrows $d * 9^k$, the next output will be a number that is strictly less for sufficiently large numbers. So, after enough iterations, the number must continually decrease until it reaches a bound region depending on the exponent. So, every sequence eventually enters a finite set of values. Since this has a finite number of values, the pigeonhole principle can be applied with the elements of the set being the holes and each output being a pigeon. So, some value must repeat eventually, therefore the sequence must eventually enter a cycle. Therefore, every starting value eventually enters a repeating structure, even though the paths and cycle lengths may largely differ.

Investigation of Higher Exponents

I extended this investigations to other exponents k, using a python program to compute the number of distinct cycles. As k increases, the function values initially grow much faster due to the digit powers increasing, but the exact same bounding arguments with pigeonhole principle ensures that all of the sequences eventually enter a finite number of outputs and must therefore cycle. However, the number and structure of the cycles themselves will become more complex as k increases.

Exponent k	Search bound	Distinct Cycles found	Cycles
$k = 2$	1000	2	(1,) (4, 16, 37, 58, 89, 145, 42, 20)
$k = 3$	1000	9	(1,) (55, 250, 133) (136, 244) (153,) (160, 217, 352) (370,) (371,) (407,) (919, 1459)
$k = 4$	1000	6	(1,) (1138, 4179, 9219, 13139, 6725, 4338, 4514) (1634,) (2178, 6514) (8208,) (9474,)
$k = 5$	1000	15	(1,) (244, 2080, 32800, 33043, 1753, 20176, 24616,

			16609, 74602, 25639, 70225, 19996, 184924, 93898, 183877, 99394, 178414, 51625, 14059, 63199, 126118, 40579, 80005, 35893, 95428, 95998, 213040, 1300) (4150,) (4151,) (8294, 92873, 108899, 183635, 44156, 12950, 62207, 24647, 26663, 23603) (8299, 150898, 127711, 33649, 68335, 44155) (9044, 61097, 83633, 41273, 18107, 49577, 96812, 99626, 133682, 41063) (9045, 63198, 99837, 167916, 91410, 60075, 27708, 66414, 17601, 24585, 40074, 18855, 71787, 83190, 92061, 66858, 84213, 34068, 41811, 33795, 79467, 101463) (10933, 59536, 73318, 50062) (24584, 37973, 93149, 119366, 74846, 59399, 180515, 39020, 59324, 63473, 26093, 67100) (54748,) (58618, 76438) (89883, 157596) (92727,) (93084,)
$k = 6$	100000	6	(1,) (17148, 383891, 1057188, 657564, 246307, 169194, 1113636, 94773, 771564, 301676, 211691, 578164, 446171, 172499, 1184692, 844403, 275161, 179996, 1758629, 973580, 927588, 1189067, 957892, 1458364, 333347, 124661, 97474, 774931, 771565, 313205) (63804, 313625) (93531, 548525, 313179, 650550) (239459, 1083396, 841700, 383890, 1057187, 513069, 594452, 570947, 786460, 477201) (282595, 824963, 845130)

(I didn't add the distinct cycles for the exponent 7 and 8 due to length issues)

For higher exponents, the bounding number of inputs to check may get very large, so computational methods may need to be expanded or adjusted. Additionally, note that f depends only on the digits of n, not their order. Since they give the same output commutatively, this reduces the number of distinct cases that must be considered for the cycles.

Algebraic Properties of Cycle Lengths

Let us now investigate the lengths of the cycles. We already investigated cycle length 1, so now we will look at what it will take for a cycle length of 2 to occur. Let us start with the exponent of 2 and two digit starting number. Since a two digit number can only output a 1 digit, 2 digit, or 3 digit number, we will split it into these cases.

1 digit: AB → C → AB, which means $A^2 + B^2 = C$ and $C^2 = 10A + B$ This gives $(A^2 + B^2)^2 = 10A + B$. This has no integer solutions, which shows that no 2 cycles exist for a one digit number.

2 digit: AB → CD → AB, which means $A^2 + B^2 = 10C + D$, and $C^2 + D^2 = 10A + B$.

Adding these, we get $A^2 + B^2 + C^2 + D^2 = 10A + B + 10C + D$, and

$A^2 - 10A + B^2 - A + C^2 - 10C + D^2 - D = 0$

$(A - 5)^2 - 25 + (B - \frac{1}{2})^2 - \frac{1}{4} + (C - 5)^2 - 25 + (D - \frac{1}{2})^2 - \frac{1}{4} = 0,$

$(A - 5)^2 + (B - \frac{1}{2})^2 + (C - 5)^2 + (D - \frac{1}{2})^2 = \frac{101}{2},$

$(2A - 10)^2 + (2B - 1)^2 + (2C - 10)^2 + (2D - 1)^2 = 202$, which is manually difficult to solve, as $(2A - 10)^2$ has a maximum value of 64, and $(2B - 1)^2$ has a maximum value of 289, which is already more than enough. Computationally, this doesn't have any digit solutions.

3 digits: AB→CDE→AB, which gives $A^2 + B^2 = 100C + 10D + E$, and $C^2 + D^2 + E^2 = 10A + B, A^2 - 10A + B^2 - B + C^2 - 100C + D^2 - 10D + E^2 - E =$ $= (A - 5)^2 + (B - \frac{1}{2})^2 + (C - 50)^2 + (D - 5)^2 + (E - \frac{1}{2})^2 = 2550.5 =$ $(2A - 10)^2 + (2B - 1)^2 + (2C - 100)^2 + (2D - 10)^2 + (2E - 1)^2 = 10202$. Through computational methods, this also does not have any digit solutions.

As the exponents get larger and the number of potential digit solutions increases, these equations become too large to bound meaningfully algebraically.

We will now investigate which cycle lengths occur for a given k.

Exponent k	Cycle lengths
$k = 2$	Length 1: 1 cycle Length 8: 1 cycle
$k = 3$	Length 1: 6 cycles Length 2: 2 cycles Length 3: 2 cycles
$k = 4$	Length 1: 4 cycles Length 2: 1 cycle Length 7: 1 cycle
$k = 5$	Length 1: 6 cycles Length 2: 2 cycles Length 4: 1 cycle Length 6: 1 cycle Length 10: 2 cycles Length 12: 1 cycle Length 22: 1 cycle Length 28: 1 cycle
$k = 6$	Length 1: 1 cycle Length 2: 1 cycle Length 3: 1 cycle Length 4: 1 cycle Length 10: 1 cycle Length 30: 1 cycle

$k = 7$	Length 1: 4 cycles Length 2: 1 cycle Length 3: 1 cycle Length 6: 1 cycle Length 12: 1 cycle Length 14: 1 cycle Length 21: 1 cycle Length 27: 1 cycle Length 30: 1 cycle Length 56: 1 cycle Length 94: 1 cycle
$k = 8$	Length 1: 3 cycles Length 25: 1 cycle Length 156: 1 cycle

For $k = 5$, the cycle lengths largely vary, ranging from numerous cycles of lengths 1 to a cycle of length 28. This is in contrast to $k = 2$, where there is only one cycle with a length of 1 and another with a length of 8. So, as the exponent increases, longer and more varied cycles appear. However, even though the behavior becomes more and more erratic, it still is confined to a finite region. This increase is not linear, and the emergence of longer cycles of length 94 and 156 suggests that the behavior becomes less predictable as k increases.

Conjecture: Even exponents tend to have less distinct cycles than odd exponents.

Argument: Observing the data shown in the table, exponents $k = 2, 3, 4, 5, 6, 7, 8$ have $2, 10, 6, 15, 6, 14, 5$ distinct cycles, respectively. The data shows a noticeable alternation between even and odd exponents. We can attempt to explain this by investigating how many distinct residues exist when even versus odd powers are taken $(mod\ 4)$. All n are $\equiv 0, 1, 2, 3\ (mod\ 4)$, so all n^2 are $0, 1, 0, 1\ (mod\ 4)$ and all $n^3 \equiv 0, 1, 0, 3$, and overall, $n^{2x} \equiv 0, 1, 0, 1\ (mod\ 4)$ because $0^{2x} \equiv 0,\ 1^{2x} \equiv 1,\ 2^{2x} \equiv 0(mod\ 4),\ 3^{2x} \equiv (4 - 1)^{2x} \equiv (-1)^{2x} \equiv 1(mod\ 4)$ and

$n^{2x+1} \equiv 0,\ 1,\ 0,\ 3\ (mod\ 4)$ due to the same logic. Since there are less distinct residues for even exponents there, there may be less residues that it can form when added together. However, this property is less useful as the number of digits increases, which leads to more distinct residues being able to be formed. So, this explanation in $(mod\ 4)$ only provides an explanation for small digit lengths. Investigating these in a higher modulus would likely provide some more rigorous explanation, but this pattern is not rigorously proven.

Observation: The most prominent cycle length is 1, which is when it is a fixed point.

Explanation: For an exponent to have a fixed point, it requires

$10^{d-1}A_1 + 10^{d-2}A_2 + \ldots 10A_{i-1} + A_i = A_1{}^k + A_2{}^k + \ldots + A_{i-1}{}^k + A_i{}^k$, or

$\sum_{s=1}^{i} 10^{d-s}A_s = \sum_{r=1}^{i} A_r{}^k$ for some integers d and k.

But, as seen in the table above, the exponent $k = 6$ does not have a fixed point other than 1. This means that this equation does not always have a solution d for a given k,

$100000A + 10000B + 1000C + 100D + 10E + F \neq A^6 + B^6 + C^6 + D^6 + E^6 + F^6$ for all digits $A,\ B,\ C,\ D,\ E,\ F$. Through computational methods, the exponent $k = 10$ also has no solutions for fixed points. For most of the exponents k, other than ones that don't have fixed points, the data shows that fixed points show up the most. Overall, fixed points are simple in structure but difficult to predict, since their existence depends on a digit power equation. This raises the question of if fixed points are so restrictive, can we determine how large a fixed point could be for a given exponent k?

Bounding the Digit Length of Fixed Points

We will now investigate the upper bound for d, the number of digits a fixed point could have for a given exponent k. The maximum number for a d digit number is $10^d - 1$ and the minimum is

10^{d-1}. The maximum value of the sum of the powers of the digits is $d * 9^k$. Roughly speaking, from these relations, it can be established that a fixed point for an exponent of k should have around k digits, leaving $d * 9^d = 10^{d-1} + 1$, as that would let the exponential terms have the same power, and the multiplier d compensates for the difference in bases. This is supported by the fact that $k = 5$ has fixed points of digit length 5. For $k = 6$, an exponent with no fixed points, the maximum is $10^6 - 1 = 999999$ and the maximum of the sum of the powers is $5 * 9^6 = 2657205$, which shows that this condition of $10^{d-1} \leq d * 9^k$ is a necessary one for fixed points, but not sufficient. Through computational approaches, if we can have a d digit number, and its decimal expansion's minimum exceeds the maximum of the sum of the powers of its digits, then it can be said that no fixed points exist for that d value and k value. Through graphing the difference $10^{x-1} - x \cdot (9^x)$, a zero occurs at ≈ 60.84, which means that the minimum value of a 61-digit number exceeds the maximum value of the sum of the 61st powers of the digits of the numbers, which fails to satisfy this inequality. However, there are some issues with this approach. Since this approach was based on the assumption that the exponent and the number of digits is the same, this pattern may not necessarily continue as the exponent increases. It is like a heuristic approach in this sense, it won't give a definite answer; As k increases, this inequality becomes harder to satisfy, explaining why fixed points occur less and less often for higher exponent values. To further restrict the possibilities, we can use modular arithmetic, which allows us to rule out numbers based on their residues.

Modular Restrictions

Another method of approaching which numbers cannot be considered for fixed points is using known impossibilities in which numbers can be expressed as the sum of exponents. For example,

no number of the form $4k + 3$ can be expressed as the sum of two squares because all numbers $n \equiv 0, 1, 2, 3 \ (mod\ 4)$, so $n^2 \equiv 0, 1, 0, 1 \ (mod\ 4)$, so two squares can never add to $3 \ (mod\ 4)$. So, this removes any two digit fixed points to consider for the exponent $k = 2$ that are of form $4k + 3$.

Additionally, for $k = 3$, for two digits, all $n \equiv 0, 1, 2, 3, 4, 5, 6 \ (mod\ 7)$, so all $n^3 \equiv 0, 1, 1, -1, 1, -1, -1 \ (mod\ 7)$, and all possible residues of the sum of two cubes are to $0, 1, 2, 6, 5 \ (mod\ 7)$. This means the two impossible residues are $3, 4 \ (mod\ 7)$. So, all two digit numbers $\equiv 3, 4 \ (mod\ 7)$ cannot be a fixed point for $k = 3$.

For three digits, using the same logic, all possible residues of the sum of three cubes are $\equiv 0, \pm 1, \pm 2, \pm 3 \ (mod\ 7)$, which covers all possible residues, so there is no modular restriction for three digits in modulo 7.

Now, let us investigate in modulo 9. For two digits, all $n \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8 \ (mod\ 9)$, so all $n^3 \equiv 0, 1, -1, 0, 1, -1, 0, 1, -1$. So, all sums of cubes are $\equiv -2, -1, 0, 1, 2 \ (mod\ 9)$, which makes the impossible residues to be: $3, 4, 5, 6 \ (mod\ 9)$. This again rules out many possible fixed points from being considered for two digits.

For three digits, it corresponds to the sum of three cubes, which can be congruent to $-3, -2, -1, 0, 1, 2, 3 \ (mod\ 9)$. So, the excluded residues are $4, 5 \ (mod\ 9)$, which rules out more possibilities for three digit fixed points for exponent $k = 3$.

The same modular argument can be used for higher exponents and higher numbers of digits to further place modular restrictions.

Restrictions on Possible Cycle lengths.

From the data collected, there are some cycle lengths that do not occur in small exponents while being relatively small cycle lengths, like lengths of 5. However, unlike the concept of fixed

points, there are no clear theoretical constraints on cycle length. So, while the apparent absence of some cycle lengths may suggest some structure, no general rule has been established and would need further research and computation.

How Numbers Move Towards Cycles

We can also investigate how different starting values move toward the cycle. For example, for fixed points, one way to get to a fixed point is to have some permutation of the digits of the fixed point. For example, $f(68, 2) = f(86, 2)$. Another way of getting to it is any permutation of the digits with any amount of zeros in between, since that would not affect the output value. This is because $0^k = 0$ for all k. For this reason, all powers of 10 go to the fixed point of 1, along with any sum of powers that can be expressed in multiple ways. As the number of digits increases, there would be more ways to get to the same number due to permutations. One can now investigate which combinations of digit powers sum to an exponent of 10. For example, $6^2 + 8^2 = 100$, which means $68,\ 100,\ 1$ is a way of getting to the fixed point of 1. This shows that different digit combinations can collapse into the same path, eventually leading to the same fixed point.

VI. Conclusions

- The function $f(n, k)$, which is defined as the sum of the kth powers of the digits of n, always enters a cycle for any starting value of n and k. This is proven through the pigeonhole principle, since repeated iteration of the function eventually repeats itself.
- Fixed points have many restrictions. By comparing the growth of n with the maximum possible value of $f(n, k)$, it can be shown that fixed points can only occur for numbers with a limited number of digits for a given exponent k.

- For small values of k, fixed points can be analyzed through algebraic and modular methods, allowing many possibilities to be ruled out.
- Computational results show that larger values of k can produce longer and more varied cycles, although the number of distinct cycles does not grow in a set increasing pattern.
- Data also shows that odd exponents tend to produce more cycles than even exponents, but this is not proven rigorously.
- All in all, fixed points have many constraints and become increasingly rare, while cycle structure becomes more complex as k increases

VII. Extensions

1. Another subject to investigate is what value of n maximizes $f(n, 2)$. The same value of n does not necessarily maximize of $f(f(n, 2), 2)$ because it would achieve a higher value depending on the size of its digits. For example, ending in a 9 would guarantee the output to be at least 81. Ending in a 89 or a 99 is also most optimal. The same stands for $f(f(f(n, 2), 2), 2)$.
2. Investigate this function for different bases. Since place values depend significantly on the base you are using, different results would stem from different bases, such as base $2,\ 4$, or 16. From researching the function so far, I expect there to be similar patterns of fixed points being rare and cycles becoming more erratic.
3. One could also explore further into the idea of the density of numbers that go to certain cycles. This is much more of a computational problem, but it would be interesting to see if there are certain biases for whether numbers have a preference for some cycle lengths like fixed points, or if there is an even distribution across all of the cycles.

VIII. References

1. Weisstein, Eric W. “Happy Number.” *MathWorld*
2. Weisstein, Eric W. “Waring’s Problem.” *MathWorld*
3. Lagarias, Jeffrey. “The 3x+1 Problem and Its Generalizations.”